Letters to the Editor—Extension of Dantzig's Algorithm to Finding an Initial Near-Optimal Basis for the Transportation Problem

Edward J. Russell,
EXTENSION OF DANTZIG’S ALGORITHM TO FINDING AN INITIAL NEAR-OPTIMAL BASIS FOR THE TRANSPORTATION PROBLEM

Edward J. Russell
Booz·Allen & Hamilton, Inc., New York, N.Y.
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This note presents a new method for finding a starting basis for transportation problems; it produces a near-optimal basis, and appears to be superior to present methods.

FOR THE Hitchcock-Koopmans transportation problem, several methods have been devised for finding initial basic feasible solutions. It is generally agreed that the time expended on finding a starting basis close to the optimal solution usually offsets the time that would be required to find the optimal solution when the starting basis is farther from the optimal solution. This note describes a new method for finding a near-optimal starting basis. The new method has two advantages: (1) Experimentation with small, but diverse, examples taken from the literature, suggests that the new method is more efficient in the sense of arriving at the optimal solution or a solution closer to the optimum than present methods. (2) The structure of the method is similar to Dantzig’s algorithm for solving transportation problems, one that is usually used in computer codes.

The transportation problem can be formulated as follows. A homogeneous product is to be shipped from m origins where quantities \(a_i (i = 1, \ldots, m)\) are available to n destinations where amounts \(b_j (j = 1, \ldots, n)\) are required. The amount shipped from the ith origin to the jth destination is \(x_{ij}\) and the corresponding unit cost is \(c_{ij}\). It is assumed that \(\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j\). The problem is to find the amounts \(x_{ij}\) so that the total transportation cost is minimized subject to origin availabilities, destination requirements, and nonnegativity constraints; that is:

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} = \text{min,} \tag{1}
\]

\[
\sum_{j=1}^{n} x_{ij} = a_i, \quad (a_i > 0, i = 1, \ldots, m) \tag{2}
\]

\[
\sum_{i=1}^{m} x_{ij} = b_j, \quad (b_j > 0, j = 1, \ldots, n) \tag{3}
\]

\[
x_{ij} \geq 0, \quad \text{for all } i, j. \tag{4}
\]
Given an initial basis, Dantzig’s algorithm repeatedly selects a new variable, \( x^*_i \), to enter the basis, until optimality is reached, by the following criterion:

\[
x^*_i = \max_{(i, j)} [(w_i + v_j - c_{ij}) > 0],
\]

where the \( u_i(i = 1, \ldots, m) \) and \( v_j(j = 1, \ldots, n) \) are the simplex multipliers associated with the \( m \) origins and \( n \) destinations, respectively.

The central idea of the new method for finding a near-optimal starting basis is to use Dantzig’s criterion, in conjunction with estimates of final (last-iteration) simplex multipliers. After experimenting with different estimates for the final simplex multipliers, we concluded that the estimates which produce the best results are:

\[
w_i = \max_j (c_{ij}); \quad (i = 1, \ldots, m) \tag{6}
\]

\[
y_i = \max_i (c_{ij}). \quad (j = 1, \ldots, n) \tag{7}
\]

The quantities \( w_i \) and \( y_j \) are the estimates for the \( u_i \) and \( v_j \) respectively. The new procedure can be described as follows:

**Step 1.** Calculate the quantities \( w_i(i = 1, \ldots, m) \) and \( y_j(j = 1, \ldots, n) \) using (6) and (7).

**Step 2.** Find the best variable, \( x^*_i \), to form a basis using the estimates found in Step 1 in Dantzig’s criterion; that is

\[
x^*_i = \max_{(i, j)} [(w_i + v_j - c_{ij}) > 0]. \tag{8}
\]
Step 3. Set the activity level of $x_{ij}^*$ equal to its minimum origin availability, $a_i^*$, or destination requirement, $b_j^*$.

Step 4. Subtract $x_{ij}^*$ from $a_i^*$ and $b_j^*$ found in Step 3. Eliminate from the transportation problem the row or column that results in a zero origin availability or destination requirement after this subtraction. Stop if all $a_i(i=1,\ldots,m)$ and $b_j(j=1,\ldots,n)$ are zero; otherwise return to Step 1.

In the absence of degeneracy, this procedure will yield a basic feasible solution in $m+n-1$ positive variables.\(^{[4]}\)

<table>
<thead>
<tr>
<th>DESTINATIONS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$w_i$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>102</td>
<td>132</td>
<td>166</td>
<td>79</td>
<td>146</td>
<td>$a_1 = 8$ [79]</td>
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<tr>
<td>2</td>
<td>130</td>
<td>96</td>
<td>96</td>
<td>167</td>
<td>170*</td>
<td>$a_2 = 7$ [96]</td>
</tr>
<tr>
<td>3</td>
<td>96</td>
<td>124</td>
<td>112</td>
<td>125</td>
<td>118</td>
<td>$a_3 = 9$ [96]</td>
</tr>
<tr>
<td>4</td>
<td>126</td>
<td>122</td>
<td>154</td>
<td>154</td>
<td>87</td>
<td>$a_4 = 3$ [87]</td>
</tr>
<tr>
<td>5</td>
<td>126</td>
<td>157</td>
<td>96</td>
<td>148</td>
<td>162</td>
<td>$a_5 = 5$ [87]</td>
</tr>
</tbody>
</table>

$\begin{align*}
b_1 &= 6 \\
b_2 &= 8 \\
b_3 &= 10 \\
b_4 &= 4 \\
b_5 &= 4
\end{align*}$

$\begin{align*}
y_j &= 96 \quad [93] \quad [96] \quad [79] \quad [87]
\end{align*}$

**Figure 2**

HOUTHAKKER's example,\(^{[4]}\) for which data were chosen at random, will be used to demonstrate the procedure. The costs $c_{ij}$ are shown in the matrix of Fig. 1 along with the corresponding $a_i$ and $b_j$. The first step of the procedure is to calculate $w_i(i=1,\ldots,m)$ and $y_j(j=1,\ldots,n)$. These quantities are shown in brackets at the right and bottom margins of Fig. 2. The quantities in the tableau of Fig. 2 are $(w_i+y_j-c_{ij})$. Using the selection criterion (8), we achieve the maximum value of 170 corresponding to $x_{25}$. In Step 3 we set $x_{25} = 4$, i.e., its destination requirement. In Step 4 we subtract $x_{25}$ from $a_5$ and eliminate column 5 from the tableau as shown in Fig. 3. The same procedure is repeated in the new tableau and we find $x_{24} = 3$. Continuing, we arrive at an initial basic feasible solution (see Fig. 4), which is close
### Figure 3

<table>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>130 96 96 167*</td>
</tr>
<tr>
<td>3</td>
<td>96 124 112 125</td>
</tr>
<tr>
<td>4</td>
<td>97 93 125 125</td>
</tr>
<tr>
<td>5</td>
<td>127 157 96 148</td>
</tr>
</tbody>
</table>

- \( \alpha_1 = 8 \)  
- \( \alpha_2 = 3 \)  
- \( \alpha_3 = 9 \)  
- \( \alpha_4 = 3 \)  
- \( \alpha_5 = 5 \)

\[ b_1 = 6 \quad b_2 = 8 \quad b_3 = 10 \quad b_4 = 4 \]

\[ y_{ij} = [96] [93] [96] [79] \]

### Figure 4

<table>
<thead>
<tr>
<th>ORIGINS</th>
<th>DESTINATIONS</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
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<tr>
<td>2</td>
<td>3 4</td>
</tr>
<tr>
<td>3</td>
<td>5 3 1</td>
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<tr>
<td>4</td>
<td>1 2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

- \( \alpha_1 = 8 \)  
- \( \alpha_2 = 7 \)  
- \( \alpha_3 = 9 \)  
- \( \alpha_4 = 3 \)  
- \( \alpha_5 = 5 \)

\[ b_1 = 6 \quad b_2 = 8 \quad b_3 = 10 \quad b_4 = 4 \quad b_5 = 4 \]
to the optimal solution. The cost of this initial solution is 1103 and is one iteration (by Dantzig’s algorithm) away from the optimal solution with a cost of 1102.

Although the example used to illustrate the new procedure is small (and hopefully unbiased) the results compare favorably with present methods for finding initial solutions. Using the same test problem, we found the cost of the initial solution by three other methods to be:

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost of the initial solution</th>
</tr>
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<tbody>
<tr>
<td>(1) Least-cost rule</td>
<td>1144</td>
</tr>
<tr>
<td>(2) Houthakker’s mutually preferred flows</td>
<td>1123</td>
</tr>
<tr>
<td>(3) Vogel’s Approximation Method</td>
<td>1104</td>
</tr>
</tbody>
</table>

In all of the test problems examined, I have found that the new method yields initial solutions that are closer to the optimal solution than current methods.

REFERENCES