MATH 153 CALCULUS I - Q U I Z II – (Section 3)

1. Calculate enough derivatives of the function $f(x) = \frac{1}{2 - 3x}$ to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.

Solution

In the following solution you write $a = 2$ and $b = -3$.

$$f(x) = \frac{1}{a + bx} = (a + bx)^{-1};$$
$$f'(x) = -b(a + bx)^{-2};$$
$$f''(x) = 2b^2(a + bx)^{-3};$$
$$f^{(3)}(x) = -3!b^3(a + bx)^{-4}.$$  

Guess: $f^{(n)}(x) = (-1)^n n! b^n (a + bx)^{-(n+1)}$  

Proof: (*) holds for $n = 1, 2, 3$

Assume (*) holds for $n = k$.

Then

$$f^{(k)}(x) = (-1)^k k! b^k (a + bx)^{-(k+1)}.$$  

So (*) holds for $n = k + 1$ if it holds for $n = k$.

Therefore, (*) holds for $n = 1, 2, 3, 4, \ldots$ by induction.

2. Find the equations of the tangent and normal lines to the curve $\tan(xy^2) = \frac{2xy}{\pi}$ at the point $\left(-\pi, \frac{1}{2}\right)$.

Solution

$$\tan(xy^2) = (2/\pi)xy$$

$$\sec^2(xy^2)(y^2 + 2xyy') = (2/\pi)(y + xy').$$

At $(-\pi, 1/2)$, $2((1/4) - y') = (1/n) - 2y'$, so

$$y' = (\pi - 2)/(4\pi(\pi - 1)).$$

The tangent has equation

$$y = \frac{1}{2} + \frac{x - 2}{4\pi(\pi - 1)}(x + \pi).$$