INVERSE FUNCTIONS

A function \( f \) is one-to-one if \( f(x_1) \neq f(x_2) \) whenever \( x_1 \) and \( x_2 \) belong to the domain of \( f \) and \( x_1 \neq x_2 \) or, equivalently, if

\[
f(x_1) = f(x_2) \implies x_1 = x_2.
\]

If a function defined on a single interval is increasing (or decreasing), then it is one-to-one.

If \( f \) is one-to-one, then it has an inverse function \( f^{-1} \). The value of \( f^{-1}(x) \) is the unique number \( y \) in the domain of \( f \) for which \( f(y) = x \). Thus,

\[
y = f^{-1}(x) \iff x = f(y).
\]

Properties of inverse functions

1. \( y = f^{-1}(x) \iff x = f(y) \).
2. The domain of \( f^{-1} \) is the range of \( f \).
3. The range of \( f^{-1} \) is the domain of \( f \).
4. \( f^{-1}(f(x)) = x \) for all \( x \) in the domain of \( f \).
5. \( f(f^{-1}(x)) = x \) for all \( x \) in the domain of \( f^{-1} \).
6. \( (f^{-1})^{-1}(x) = f(x) \) for all \( x \) in the domain of \( f \).
7. The graph of \( f^{-1} \) is the reflection of the graph of \( f \) in the line \( x = y \).

A function \( f \) is self-inverse if \( f^{-1} = f \), that is, if \( f(f(x)) = x \) for every \( x \) in the domain of \( f \).

Derivatives of Inverse Functions

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.
\]

In Leibniz notation we have

\[
\frac{dy}{dx} \bigg|_{y=f^{-1}(x)} = \frac{1}{\frac{dx}{dy} \bigg|_{y=f^{-1}(x)}}.
\]
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EXPONENTIAL AND LOGARITHM FUNCTIONS

Exponentials

An exponential function is a function of the form \( f(x) = a^x \), where the base \( a \) is a positive constant and the exponent \( x \) is the variable.

If \( a > 1 \), then \( a^x \) is an increasing function of \( x \); if \( 0 < a < 1 \), then \( a^x \) is decreasing.

\[
\begin{align*}
\text{If } a > 1, & \quad \lim_{x \to -\infty} a^x = 0 \quad \text{and} \quad \lim_{x \to \infty} a^x = \infty, \\
\text{If } 0 < a < 1, & \quad \lim_{x \to -\infty} a^x = \infty \quad \text{and} \quad \lim_{x \to \infty} a^x = 0.
\end{align*}
\]

Graphs of some exponential functions

Logarithms

If \( a > 0 \) and \( a \neq 1 \), the function \( \log_a x \), called the logarithm of \( x \) to the base \( a \), is the inverse of the one-to-one function \( a^x \):

\[
y = \log_a x \quad \iff \quad x = a^y, \quad (a > 0, \quad a \neq 1).
\]

\[
\log_a (a^x) = x \quad \text{for all real } x \quad \text{and} \quad a^{\log_a x} = x \quad \text{for all } x > 0.
\]

Laws of logarithms

If \( x > 0, y > 0, a > 0, b > 0, a \neq 1, \) and \( b \neq 1 \), then

(i) \( \log_a 1 = 0 \)  \hspace{1cm} (ii) \( \log_a (xy) = \log_a x + \log_a y \)

(iii) \( \log_a \left( \frac{1}{x} \right) = -\log_a x \)  \hspace{1cm} (iv) \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \)

(v) \( \log_a (x^y) = y \log_a x \)  \hspace{1cm} (vi) \( \log_a x = \frac{\log_b x}{\log_b a} \)
The Natural Logarithms

If $x > 0$, then

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

(i) $\ln(xy) = \ln x + \ln y$

(ii) $\ln \left(\frac{1}{x}\right) = -\ln x$

(iii) $\ln \left(\frac{x}{y}\right) = \ln x - \ln y$

(iv) $\ln \left(x^r\right) = r \ln x$

$$\lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to 0^+} \ln x = -\infty.$$
The Exponential Function

\[ e^x = \exp x \quad \text{for all real } x. \]

\[ e = 2.71828 \ 1828 \ 45 \ 90 \ 45 \ldots \]

\[ y = \exp x \iff x = \ln y \quad (y > 0). \]

\[ \lim_{x \to -\infty} e^x = 0, \quad \lim_{x \to \infty} e^x = \infty. \]

\[ \ln x = \log_e x. \]

The General Exponentials and Logarithms

The general exponential \( a^x \)

\[ a^x = e^{x \ln a}, \quad (a > 0, \ x \ \text{real}). \]

\[ \frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a. \]

\[ \frac{d}{dx} \log_a x = \frac{1}{x \ln a}. \]